Re-Examining the Reported Rates of Return to Food and Agricultural Research and Development: Comment

James F. Oehmke

Hurley, Rao, and Pardey (2014) argue to replace the internal rate of return with the modified internal rate of return for the evaluation of public research investment. The crux of their argument is that the internal rate of return "assumes intermediate cash flows can be reinvested (or borrowed) at same return as the initial investment, which is generally not correct or reasonable," (page 1492). This article first demonstrates that reinvestment decisions are embodied in the project specification, and that the internal rate of return makes no inherent reinvestment assumption. The article then clarifies the algebraic properties of the marginal internal rate of return and the reinvestment implications of the internal rate of return and modified internal rate of return within the context of public agricultural research evaluation.

Key words: Agricultural research, benefit-cost ratio, internal rate of return, modified internal rate of return, rates of return on R&D.

JEL codes: G31, Q16, Q18.

Hurley, Rao, and Pardey (2014) have disavowed over 50 years of research investigating the internal rate of return (IRR) to public agricultural research and development, arguing to replace the IRR with the modified internal rate of return (MIRR). The crux of these authors' argument is the following statement: "[t]he IRR assumes intermediate cash flows can be reinvested (or borrowed) at same return as the initial investment, which is generally not correct or reasonable," (Hurley, Rao, and Pardey 2014). Hurley, Rao, and Pardey then seek an alternative measure.[1] The MIRR contains an interest rate representing the cost of capital and one representing returns on reinvestment, which in the authors' perspective largely justifies replacing the IRR with the MIRR. The result is that the returns to public agricultural research as measured by the MIRR look much lower than those in the literature, and funding levels appear close to optimal.

The finance and project evaluation literature provide limited support for Hurley, Rao, and Pardey: although some authors prefer the MIRR on the assumption that the IRR requires reinvestment (e.g., Anderson and Barber 1994; Kierulff 2008), the literature also explicitly rejects the idea that the IRR actually does require a reinvestment assumption (Alchian 1955; Hirshleifer 1958, 1959;[2]

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[1] Alternative explanations include high marginal tax costs (Dahlby 2008; Fox 1985) (Hurley, Rao, and Pardey (2014) adjust their MIRR calculations for the marginal excess burden of taxes, but not their IRR calculations, so it is difficult to determine if reported IRRs to research would be "credible" (Hurley, Rao, and Pardey 2014) if tax burdens were included), government myopia (Oehmke 1986), political preferences (Oehmke and Yao 1990), the modal IRR (not the mean) indicates much smaller investment gaps (Roseboom 2002), limited research options in developing countries Roseboom 2002 and alternative objective function criteria (Gardner and Lesser 2003; Arovuori 2008), among others.

[2] Hirshleifer (1958), cited by Hurley, Rao, and Pardey (2014) as saying that the IRR has an implicit reinvestment decision, actually stated the contrary: "If, of course, we use some external discounting rate (for example, the cost of capital or the rate of an outside lending opportunity), we will be departing from the idea of a purely internal growth rate," (p. 347, parentheses in original). Hirshleifer (1958) did argue that an optimal portfolio potentially includes the use of investment in other projects, borrowing at an external rate, and lending at an external rate—but not that the IRR implicitly assumes reinvestment. A year later Hirshleifer (1959) concluded that the "… internal rate concept can be used correctly, but only after a more sophisticated redefinition, as provided especially in Bailey's forth-coming work." Bailey's "forthcoming" work (Bailey 1959) explicitly considered withdrawals to be removed and not reinvested.

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Doenges 1972; Dudley 1972; Dorfman 1981; Beidleman 1984; Keef and Roush 2001; Johnston, Forbes, and Hatem 2002; Ross, Westerfield, and Jordan 2013; Walker, Check, and Randall 2011; Cheremushkin 2012; Rich and Rose 2014). Rich and Rose (2014) argue that the IRR has "neither an implied distortion of the intermediate cash flows nor an implicit rate of return on any reinvested cash flows … finance textbooks should disavow any implicit reinvestment rate assumption in the IRR technique."

Moreover, the MIRR has algebraic characteristics that may make it unpalatable for a number of uses, including the evaluation of public agricultural research investments.

This article clarifies the algebraic properties of the MIRR and the reinvestment implications of the IRR and MIRR within the context of public agricultural research evaluation. It first provides a discussion of some issues with the algebra of the MIRR that inform both the reinvestment issue and the application of MIRRs to research. The article then takes up the reinvestment question explicitly, and finally draws conclusions.

Some Algebraic Properties of IRR and MIRR

A project i is a stream of benefits and costs: $P_i \equiv \{b_{i,0}, \ldots b_{i,T,} c_{i,0}, \ldots c_{i,T}\}$, where the first subscript indexes the project and the second indexes the date; when only one project is of interest, the initial subscript will be suppressed (as in Hurley, Rao, and Pardey 2014). Further, $\wp \equiv \{P_i \mathbb{R}_+^{2T}$ is the set of feasible projects of length T. A valuation is a function V from the set of projects to the real line: $V : \wp \rightarrow \mathbb{R}$. The IRR is a valuation function, whose solution is defined to be that value of r which solves Hurley, Rao, and Pardey's equation (1). The *MIRR* is also a valuation function, defined according to Hurley, Rao, and Pardey's equations (2) or (3).[3]

The literature does not reach consensus on a single set of desirable features for valuation functions, perhaps because they are used for a variety of purposes, and what is a desirable feature for one application may not be desirable for a different application. However, it is of interest to explore some possibly desirable features and ascertain whether the IRR and MIRR have these features.

Optimality

The MIRR contains an assumption of suboptimal reinvestment behavior by assuming that benefits accruing at time $t<T$ are reinvested at rate δ^r rather than used to repay existing debt accruing interest at rate $\delta^c > \delta^r$. This is relevant to agricultural R&D, where typically investments and benefits (from prior investment) occur simultaneously.

For any $\delta^c > \delta^r$ there are projects that the MIRR calculated under reinvestment will reject ($MIRR_{reinvest} < \delta^c$) but the MIRR under optimal repayment behavior will accept ($MIRR_{repay} > \delta^c$). That is, the ranking of projects can be reversed by the standard MIRR assumption of suboptimal reinvestment behavior (proofs appear in the appendix).

Since the IRR does not make a reinvestment assumption (see below), there is no assumption of sub-optimal reinvestment behavior.

Invariance to Non-material Changes

Valuation measures could be invariant to non-material changes in the project being evaluated. Define the null project $P_0 \equiv \{b_{i,0} \equiv 0, \ldots b_{i,T} \equiv 0, c_{i,0} \equiv 0, \ldots c_{i,T} \equiv 0\}$ to be the project of duration T with benefits and costs identically equal to zero. Valuation measures may or may not be invariant to the addition of a null project to the existing project. That is, $V(P_{1,\ 0}) \stackrel{?}{=} V(P_{1,\ 0} + P_{0,\ s})$, where the second subscript indexes the project start date. Invariance to non-material changes says that the addition of a set of zeroes does not change the valuation of the project.

The IRR is generically invariant to non-material changes. The MIRR is invariant so long as the non-material changes are made within the original time frame. However, adding non-material items that extend the time frame unambiguously reduces the MIRR even though the dates of all material costs and benefits are unchanged (see appendix).

Invariance with Respect to Replication

Project $P_{i,0}$ is said to be replicable at date s if $P_{i,s} \in \wp$. Viewing agricultural research as a recurrent process in essence is replicating the

[3] In the MIRR there is explicit reinvestment of benefits at an external reinvestment rateδ^r. The Excel MIRR formula differs from the Hurley, Rao, and Pardey (2014) formula in that at time $t<T$, Excel reinvests net proceeds b_t-c_t and Hurley, Rao, and Pardey reinvest gross benefits, b_t. The reinvestment rate is typically assumed to be lower than the project's cost of funds or borrowing rate, $\delta^r < \delta^c$; this assumption is maintained throughout.

research project on an annual or periodic basis, which corresponds to most research budgeting processes. This is also one interpretation of "reinvestment" in agricultural research—perhaps literally for private sector research where today's funding source is the returns from earlier research, and figuratively for public sector research where benefits of prior research justify the expenditure of today's tax dollars on more research. In this case we can examine whether the valuation satisfies $V(P_{i,0}) = V(P_{i,0}+P_{i,s})$. This equation says that the valuation of a project starting at time 0 is equal to the valuation of the same project starting at time 0 plus an exact replication starting at time *s*. This is intuitively compelling for valuations expressed as rates since the cost and benefit flows are the same for the replicated project as for the original project. It does not make sense for valuations in levels such as the net present value (NPV) since benefits and costs change for the project plus replication.

The IRR is invariant with respect to replication; the MIRR is not. The replication of a successful project P_1 with the replication starting at a date s > 0 unambiguously lowers the *MIRR*. That is, $MIRR(P_{1,0}) > MIRR(P_{1,0}+P_{1,s},\ s > 0)$ (see appendix).

Project Dominance

The P_j is defined to strongly dominate P_i(P_j *s.d.* P_i) *if* $\forall t\ c_{i,t} \leq c_{j,t}$ *and* $b_{i,t} \geq b_{j,t}$, with at least one strict inequality. That is, P_2 *s.d.* P_1 if P_2 costs are never higher and benefits are never lower than P_1, with at least one cost being strictly lower or benefit strictly larger. A useful property of a valuation measure is that a dominating project has a higher value than the dominated project: $P_2\ s.d.\ P_1 \Rightarrow V(P_2) > V(P_1)$.

The IRR respects project dominance. The MIRR does so when the increased benefit (for example) occurs in the original time frame, but may not if the increased benefit occurs outside of the original time frame. For example, extending project *Pi* by accruing a small additional benefit at time $T+1$ could result in a lower *MIRR* (see appendix).

The MIRR violates project dominance and the invariance properties because of the peculiar role of the endpoint time *T* in relation to the MIRR reinvestment assumption. In the calculation, benefits b_t accruing at time $t < T$ are compounded forward to time *T* at rate δ^r and then discounted back to time 0 at the *MIRR*. This effectively diminishes the implicit value of b_t by $(MIRR - \delta^r)$% for each of the *T-t* periods in which this compounding forward and then discounting backwards is applied. For example, the MIRR calculation values a $1 cost at time zero at $1, but a $1 benefit at time 0 at $(1+\delta^r)^T/(1+MIRR)^{\mathrm{T}} < 1$ (when $MIRR > \delta^r$). This implicit "double-discount" factor on benefits accrued at time *t* will vary with T, δ^r, and the *MIRR*. The larger the difference between *MIRR* and δ^r, the greater the discount factor, so that benefits from the most successful projects are discounted the most. The larger the *T* (i.e., the larger is *T-t ceteris paribus*), the greater the implicit diminution of b_t. This is very different from saying that with longer projects benefits further in the future are more heavily discounted. Instead, it means that *benefits at any fixed point in time including time 0 are discounted more if the project length T is larger.*

To demonstrate this point algebraically, define $MBCR \equiv FVB(\delta^B)/PVC(\delta)$ and hold *MBCR* constant.[4] Substituting *MBCR* into Hurley, Rao, and Pardey's equation (2) and taking limits reveals $lim_{T\to\infty}\ MIRR(T) \equiv lim_{T\to\infty} \sqrt[T]{MBCR} - 1 = 0\ for\ MBCR > 0$. In other words, for a constant *MBCR*, simply picking a large enough *T* will drive the *MIRR* to 0. It is particularly anomalous that whether a project has a "favorable" $MBCR > 1$ or an "unfavorable" $MBCR < 1$, the *MIRR* still tends to 0. This MIRR characteristic is particularly important to projects and policy decisions with long lags between investment and returns, including public agricultural research that has lags as long as 50 years (Alston et al. 2011) and policies such as conservation reserves or agricultural carbon contracting (c.f. Gulati and Vercammen 2005).

Cost of Capital

Intuitively, we might think that increases in the cost of capital reduce the project valuation ($\partial V/\partial \delta^c < 0$) (e.g., NPVs are usually increasing in the cost of capital) and/or the

[4] Replicating a project over time does not hold MBCR constant, but results in similarly unappealing behavior because the replication increases T. For example, the project with c≡1, b ≡2, for all t, 0<=t<=T, MIRR(2%, 3%) falls as T increases and numerically approaches a seeming limit of 2%—for T = 5 the MIRR is 17.7% and for T = 1,000,000 the MIRR is fractionally over 2.00%. This raises the question of whether a 2% return is a reasonable characterization of the returns from a project that doubles investment every year for a million years (thanks to an anonymous referee for this example).

difficulty of passing the investment test (e.g., as in the IRR criterion $V > \delta^c$).

The IRR is consistent with the increasing difficulty of the investment test.

The MIRR is not consistent because it is increasing in the borrowing rate whenever there exists $t > 1$ such that $C_t > 0$, *ceteris paribus*. That is, the MIRR is increasing in the borrowing rate except when all project costs are paid up front. This is seen in Hurley, Rao, and Pardey's figure 2, with downward sloping isoquants, indicating that to hold MIRR constant a higher (lower) borrowing rate is associated with a lower (higher) reinvestment rate. It can also be seen directly from Hurley, Rao, and Pardey's equation (2), where the MIRR can be seen to be increasing in δ^c. Normally, a higher borrowing cost would be expected to lower the project value, but with the MIRR the opposite is generically true.

Note that there is a peculiarity to the IRR formula in that it is possible to have multiple solutions (which has been a critique of the IRR: see, e.g., Robison and Barry 1998)—and in these cases the NPV will have a specific region(s) in which a higher discount rate increases the calculated project NPV (Oehmke 2000).

Reinvestment

We now turn attention explicitly to reinvestment assumptions and practices in the IRR. We first show that the IRR algebra does not depend on a reinvestment assumption. Perhaps the simplest approach in the literature is to track the financial balance sheet of the investment project. The accumulated net project assets are defined recursively:

$$(1) \quad I_t \equiv (1+r)I_{t-1} + c_t - b_t, \; for \; t = 0, \ldots T; \; I_{-1} \equiv 0.$$

Equation (1) means that this period's assets equal last period's assets, plus capital appreciation at rate r, plus any new investment c_t, less any payouts b_t. This corresponds to generally accepted accounting procedures and to Bailey's (1959) concept that the continuing investment in the project equals gross proceeds $(1+r)I_{t-1}$ less net withdrawals $(b_t - c_t)$. The key point is that b_t is returned to the investor and thus there is no reinvestment assumption made: I_t is independent of whether the beneficiary consumes the benefit or invests it externally. Reinvestment in the project is covered by the c_t term, including the possibilities that c_t=0, indicating no reinvestment and that $c_t = b_t$, indicating "full" reinvestment; it is, however, sensitive to changes in the reinvestment assumption, that is, changing from c_t= 0 to $c_t = b_t > 0$ changes I_t.

Changing project costs from c_t= 0 to $c_t = b_t > 0$ is a material change in benefits and costs and thus by definition is a different project. Thus, the assumption that the **value** of the IRR function does not change with changes in reinvestment is incorrect. The relevant outcome is rather: the IRR function can evaluate a project with reinvestment or one without—the reinvestment assumption is in the project specification, not the IRR function itself.

To see the relationship between I_t and *IRR*, note that by mathematical induction

$$(2) \quad I_T = \sum_{s=0}^{T} (1+r)^{T-s}(c_s - b_s) = -(1+r)^T \sum_{s=0}^{T} \frac{b_t - c_t}{(1+r)^s}.$$

When $r = IRR$, then the last summation is 0 and so $I_T = 0$ and conversely for $r \neq -1$. In other words, the IRR formula does not make a reinvestment assumption but is algebraically consistent with whatever re-investment profile is specified in the c_t terms. It is again important to note that changing the reinvestment decision changes the c_t terms and will therefore change the estimated *IRR*.

A reinvestment critique of the IRR arises when comparing project P_1 of duration T_1 with project P_2 of duration $T_2 > T_1$ (with the same initial investment levels) and questioning what happens to the payout from P_1 in the period $(T_1, T_2]$. Consider the reinvestment portfolio of P_1 plus investing in a risk-free asset from time T_1 to T_2, $R_{T1,T2} \in \wp$. It is easy to construct examples where $IRR(P_1) > IRR(P_2)$ but the time T_2 cash value is larger for P_2 than for $P_1 + R_{T1,T2}$. The simple but inappropriate comparison of $IRR(P_1)$ with $IRR(P_2)$ gives the wrong ranking. The appropriate comparison is between $IRR(P_1 + R_{T1,T2})$ and $IRR(P_2)$, which provides the correct ranking (see appendix).

Heuristically, the critique's essence is that longevity of returns (even discounted) is valuable: comparing P_1 with P_2 directly ignores P_2's longevity. The MIRR is decreasing in T, also diminishing the value of longevity. It is even possible that $MIRR(P_2)$ would increase if P_2 were abbreviated to end at time T_1 (proved in the project dominance discussion). Thus, the MIRR does not necessarily solve this reinvestment issue.

Conclusion

The Hurley, Rao, and Pardey criticism that the reported *IRR*s are too high because the IRR suffers from an implausible reinvestment assumption does not hold up analytically. The IRR formula is indifferent as to whether the investor consumes returns or invests them in a different project. Different reinvestment assumptions by definition change a project, and the *IRR* value must be recalculated based on the new costs and benefits. As projects differ with maintained reinvestment assumptions, so do their estimated *IRR*s. The *IR*R for the portfolio of the initial project plus the reinvestment project(s) gives an accurate ranking.

The MIRR compares portfolios of borrowing, project investment, and reinvestment in a single formula, but at the cost of suboptimal reinvestment behavior that can reverse project rankings, and in a way that does not solve all reinvestment issues. The MIRR is decreasing in project duration (T). It double-discounts project benefits in a way that increases with project success($MIRR - \delta^r$). Since agricultural research is typically a successful project with long-lived benefits, application of the MIRR to agricultural research is subject to these criticisms.

The conclusion is that the IRR should be retained as a valid measure of returns to public agricultural research.

References

Alchian, A.A. 1955. The Rate of Interest, Fisher's Rate of Return over Costs and Keynes' Internal Rate of Return. *American Economic Review* 45 (5): 938–43.

Alston, J.M., M.A. Andersen, J.S. James, and P.G. Pardey. 2011. The Economic Returns to U.S. Public Agricultural Research. *American Journal of Agricultural Economics* 93 (5): 1257–7.

Anderson, G.A., and J.R. Barber. 1994. Project Holding-Period Rate of Return and the MIRR. *Journal of Business Finance & Accounting* 21 (4): 613–8.

Arovuori, K. 2008. Controversies between Stated Agricultural Policy Objectives and Policy Measures in the EU's CAP. 12th Congress of the European Association of Agricultural Economists – EAAE 2008.

Bailey, M.J. 1959. Formal Criteria for Investment Decisions. *Journal of Political Economy* 67 (5): 476–88.

Beidleman, C.R. 1984. Discounted Cash Flow Reinvestment Rate Assumptions. *Engineering Economist* 29 (2): 127–39.

Cheremushkin, S.V. 2012. There is No Hidden Reinvestment Assumption in Discounting Formula and IRR: Logical and Mathematical Arguments. Unpublished manuscript. Available at: http://ssrn.com/abstract=1982828 or http://dx.doi.org/10.2139/ssrn.1982828.

Dahlby, B. 2008. *The Marginal Cost of Public Funds: Theory and Applications*. Cambridge, MA: MIT Press.

Doenges, R.C. 1972. The Reinvestment Rate Problem in a Practical Perspective. *Financial Management* 1 (1): 85–91.

Dorfman, R. 1981. The Meaning of Internal Rate of Return. *Journal of Finance* 36 (5): 1011–21.

Dudley, C.L., Jr. 1972. A Note on Reinvestment Assumptions in Choosing between Net Present Value and Internal Rate of Return. *Journal of Finance* 27 (4): 907–15.

Fox, G. 1985. Is the United States Really Underinvesting in Agricultural Research? *American Journal of Agricultural Economics* 67 (4): 806–12.

Gardner, B., and W. Lesser. 2003. International Agricultural Research as a Global Public Good. *American Journal of Agricultural Economics*. 85 (3): 692–7.

Gulati, S., and J. Vercammen. 2005. The Optimal Length of an Agricultural Carbon Contract. *Canadian Journal of Agricultural Economics* 53 (4): 359–73.

Hirshleifer, J. 1958. On the Theory of Optimal Investment Decision. *Journal of Political Economy*. 66 (4): 329–52.

———. 1959. Marginal Efficiency of Capital: Comment. *Economic Journal* 65 (275): 592–3.

Hurley, T.M., X. Rao, and P.G. Pardey. 2014. Re-examining the Reported Rates of Return to Food and Agricultural Research and Development. *American*

Journal of Agricultural Economics 96(5): 1492–504.
Johnston, K., S. Forbes, and J.J. Hatem. 2002. Reinvestment Rate Assumptions in Capital Budgeting: A Note. *Journal of Economics and Finance Education* 1 (2): 28–9.
Keef, S.P., M.L. Roush. 2001. Discounted Cash Flow Methods and the Fallacious Reinvestment Assumption: A Review of Recent Texts. *Accounting Education: An International Journal* 10 (1): 105–16.
Kierulff, H. 2008. MIRR: A Better Measure. *Business Horizons* 51 (4): 321–9.
Oehmke, J.F. 1986. Persistent Underinvestment in Agricultural Research. *Agricultural Economics* 1 (1): 53–65.
Oehmke, J.F., and X.B. Yao. 1990. A Policy Preference Function for Government Intervention in the U.S. Wheat Market. *American Journal of Agricultural Economics* 72 (3): 631–40.
Oehmke, J.F. 2000. Anomalies in Net Present Value Calculations. *Economics Letters* 67 (3): 349–51.
Rich, S.P., and J.T. Rose. 2014. Re-examining an Old Question: Does the IRR Method Implicitly Assume a Reinvestment Rate? *Journal of Financial Education* 40 (1/2): 152–66.
Robison, L.J., and P.J. Barry. *Present Value Models and Investment Analysis*. 1998. East Lansing, MI: Michigan State University Press.
Roseboom, J. 2002. Underinvestment in Agricultural R&D Revisited. *Quarterly Journal of International Agriculture* 41 (4): 297–316.
Ross, S.A., R.W. Westerfield, and B.D. Jordan. 2013. *Essentials of Corporate Finance*, 8th Ed. New York, NY: McGraw-Hill/Irwin.
Walker, J.S., H.F. Check, Jr., and K.L. Randall. 2011. Does the Internal Rate of Return Calculation Require a Reinvestment Rate Assumption? – There Is Still No Consensus. West Chester University: Pennsylvania Economic Association Proceedings Annual Conference, June 3–5, 2010, 118–130. Available at: http://aux.edinboro.edu/pea/pub/Proceed2010/PEA2010ConferenceProceedings.pdf.

Appendix

This appendix contains proofs of the various claims made.

Optimality & MIRR Ranking Reversal

For simplicity, consider the project with benefit-cost stream $\begin{bmatrix} 0 & b_1 & b_2 \\ c_0 & c_1 & 0 \end{bmatrix}$, with $b_1 < c_1$; this is easily generalized to project with longer intermediate streams of benefits and costs. The MIRR of this project is $\left(\frac{b_1\ (1+\delta^r)+\ b_2}{c_0+\ \frac{c_1}{(1+\delta^C)}}\right)^{\frac{1}{2}} - 1$ under the standard MIRR assumption that b_1 is reinvested rather than used to pay down project debt.

Consider the alternative that b_1 is used to pay down project debt at time 1, which is $c_1(1+\delta^c)$. In this case the MIRR is $\left(\frac{b_2}{c_0+\ \frac{c_1-\ b_1}{1+\delta^c}}\right)^{\frac{1}{2}} - 1$. Note that when $b_1 = 0$ there is no material reinvestment decision and the MIRRs are equal; assume $b_1 > 0$. A ranking reversal occurs when MIRR(reinvestment) << MIRR(repayment), or

$$\text{(A.1)}\quad \frac{b_1\ (1+\delta^r)+\ b_2}{c_0+\ \frac{c_1}{(1+\delta^c)}} < (1+\delta^c)^2 < \frac{b_2}{c_0+\ \frac{c_1-\ b_1}{1+\delta^c}}.$$

Dividing all expression by $(1+\delta^c)^2$ results in

$$\text{(A.2)}\quad \frac{b_1\ (1+\delta^r)+\ b_2}{c_0(1+\delta^c)^2+\ c_1(1+\delta^c)} < 1 < \frac{b_2}{c_0(1+\delta^c)^2+(c_1-\ b_1)(1+\delta^c)}.$$

The first inequality holds when

$$\text{(A.3)}\quad b_1\ (1+\delta^r) < c_0(1+\delta^c)^2 + c_1(1+\delta^c) - b_2.$$

The second inequality holds when

$$\text{(A.4)}\quad b_1\ (1+\delta^c) > \ c_0(1+\delta^c)^2 + \ c_1(1+\delta^c) - \ b_2.$$

The difference between the two inequalities is in the coefficient on b_1: in suboptimal reinvestment the compounding factor on b_1 is $(1+\delta^r)$; in optimal repayment the compounding factor is$(1+\delta^c) > \ (1+\delta^r)$. Clearly, the inequalities cannot be satisfied simultaneously when $b_1 = 0$ or when $\delta^r = \ \delta^c$.

To construct an example of ranking reversal when $b_1 > 0$ and $\delta^r \ < \ \delta^c$, choose

$$\text{(A.5)}\quad b_1 = \frac{c_0(1+\delta^c)^2 + \ c_1(1+^c) - \ b_2}{(1+\delta^c)} + \epsilon,$$

$$\epsilon > 0.$$

Then the second inequality holds. Substituting the expression for b_1 into the first inequality results in

$$\text{(A.6)}\quad \frac{(1+\delta^r)}{(1+\delta^c)}\left(c_0(1+\delta^c)^2 + \ c_1(1+\delta^c) - \ b_2\right) + (1+\delta^r)\epsilon \ < \ c_0(1+\delta^c)^2 + c_1(1+\delta^c) - b_2.$$

The first time on the left-hand side is strictly less than the right-hand side. The second term on the left-hand side goes to 0 as $\epsilon \to 0$. Consequently, for small enough $\epsilon > 0$, the inequalities are simultaneously satisfied, demonstrating ranking reversal.

Non-material Changes

Let P_1 be a project of length T and consider the addition of $(b_{T+1} = 0, \ c_{T+1} = 0)$ to the project as a benefit and cost at time T+1. Denote the two MIRRs by $MIRR(T)$ and M $IRR(T+1)$, respectively. For simplicity, let $FVB \ \equiv \ \left(\sum_{t=0}^{T} b_t(1+\delta^r)^{T-t}\right)$ and $PVC \ \equiv \left(\sum_{t=0}^{T} c_t(1+\delta^c)^{-t}\right)$ be the future value of benefits and the present value of costs, respectively. Then

$$\text{(A.7)}\quad MIRR(T) \equiv \left(\frac{FVB}{PVC}\right)^{\frac{1}{T}} - 1,$$

and we assume

$$\text{(A.8)}\quad \left(\frac{FVB}{PVC}\right)^{\frac{1}{T}} > \ 1+\delta^c > 1+\delta^r.$$

That is, project P_1 is successful by the MIRR criterion. Then

$$\text{(A.9)}\quad MIRR(T+1) = \left(\frac{FVB(1+\delta^r)+0}{PVC+0}\right)^{\frac{1}{T+1}}$$
$$= \left(\frac{FVB}{PVC}\right)^{\frac{1}{T}} \left(\frac{1+\delta^r}{\left(\frac{FVB}{PVC}\right)^{\frac{1}{T}}}\right)^{\frac{1}{T+1}}$$
$$-1 < \left(\frac{FVB}{PVC}\right)^{\frac{1}{T}} - 1$$
$$= MIRR(T)$$

where the inequality holds because P_1 is successful. Therefore, the non-material change that extends the life of the project from T to T+1 unambiguously decreases the MIRR.

Replication

For simplicity, we deal with the case in which the original project $P_{1,0}$ runs from time $t=0$ to time $t=T$ and then is replicated at time $s \geq 0$, with the replicated project running from time $t=s$ to $T+s$. We also assume that MIRR($P_{1,0}$)>δ^c.

a. IRR. It is straightforward to show that the IRR is invariant to replication of this form, i.e. IRR($P_{1,0}$) = IRR($P_{1,0} + P_{1,s}$).
b. MIRR

For given δ^c and δ^r, let F≡ FVB(δ^r) be the future value at time T of $P_{1,0}$ benefits, and P≡ PVC(δ^c) be the present value at time 0 of $P_{1,0}$ costs, following the Hurley, Rao, and Pardey (2014) terminology in equation (2) on p. 4. Then, $MIRR(P_{1,0}) = \sqrt[T]{\frac{F}{P}} - 1$. Now consider replicating the project from time $t=s$ to $T+s$, so that the evaluation period is now 0 to $T+s$. The MIRR of the joint project (original plus replication) is expressed as

$$\text{(A.10)}\quad MIRR\left(P_{1,0}+\ P_{1,s}\right) = \sqrt[T+s]{\frac{(1+\delta^r)^s\ F+F}{P+(1+\delta^c)^{-s}\ P}}\ -\ 1.$$

The first term in the numerator of the radical represents the future value of the benefits from the original project, accumulated forward from time $t=T$ to $T+s$ at rate δ^r . The second term in the numerator is the future value at time $T+s$ of benefits accrued by the replicated project from time s to $T+s$ (and equals the future value at time T of benefits accrued under the original project from time $t=0$ to $t=T$). The first term in the denominator represents the present value at time $t=0$ of costs accrued from time $t=0$ to $t=T$ of the original portion. The second term in the denominator represents the present value at time $t=T$ of costs accrued from time $t=$s to $T+s$ of the replicated portion of the project, discounted from time $t=s$ to time $t=0$ by the discount factor $(1+\delta^c)^{-s}$.

Combining terms results in

$$\begin{aligned}\text{(A.11)}\quad & MIRR\left(P_{1,0}+\ P_{1,s}\right)\\ &= \sqrt[T+s]{\frac{F((1+\delta^r)^s+1)}{P((1+\delta^c)^s+1)}(1+\delta^c)^s}\ -1\\ &= \sqrt[T+s]{\frac{F}{P}}\ \sqrt[T+s]{\frac{((1+\delta^r)^s+1)}{((1+\delta^c)^s+1)}(1+\delta^c)^s}\\ &\quad -1\ <\ \sqrt[T+s]{\frac{F}{P}}\\ &\quad \times\sqrt[T+s]{\frac{((1+\delta^c)^s+1)}{((1+\delta^c)^s+1)}(1+MIRR(P_{1,0}))^s}\\ &\quad -1\ =\ \sqrt[T+s]{\frac{F}{P}}\ \sqrt[T+s]{\left(\frac{F}{P}\right)^{\frac{s}{T}}}\ -1\end{aligned}$$

where the inequality holds since $\delta^r < \delta^c <$ MIRR($P_{1,0} = P_{1,s}$). But the final right-hand side is simply $MIRR(P_{1,0})$. Hence, $MIRR\left(P_{1,0}+\ P_{1,s}\right) < MIRR\left(P_{1,0}\right)$. That is, the MIRR for the joint project with the original portion and the replicated portion is unambiguously lower than the MIRR for the original project by itself.

Project Dominance

We have already seen that the MIRR is sensitive to the addition of non-material terms that extend the life of the project. Proving that MIRR breaks the project dominance criterion is simply a matter of extending the life of the project by adding terms that are material but small in relative terms. First, note that for a project of length T, increasing any benefit b_t or decreasing any cost c_t, $0<=t<=T$ will result in a larger MIRR, as desired. The issue comes with adding benefits or costs that change the duration of the project.

For simplicity we deal with the case in which there is a single additional positive benefit in year $T+1$, $b_{T+1}>0$, with no other changes to the project. The new MIRR is

$$\text{(A.12)}\quad MIRR\ (P_1+b_{T+1}) = \sqrt[T+1]{\frac{FVB(\delta^r)(1+\delta^r)+\ b_{T+1}}{PVC(\delta^c)}}-1.$$

This will exceed the previous MIRR if

$$\begin{aligned}\text{(A.13)}\quad & \frac{FVB(\delta^r)(1+\delta^r)+\ b_{T+1}}{PVC(\delta^c)}\\ &> \left(\frac{FVB(\delta^r)}{PVC(\delta^c)}\right)^{1+\frac{1}{T}}\ or\ \frac{b_{t+1}}{FVB(\delta^r)}\\ &> \left(\left(\frac{FVB(\delta^r)}{PVC(\delta^c)}\right)^{\frac{1}{T}}-1\right)-\delta^r.\end{aligned}$$

The left-hand side is the percentage increase in benefits b_{t+1} as a proportion of the time T value of all the preceding benefits. The first term on the right-hand side is the MIRR for the original project—the discount rate that is applied to existing benefits if the project is extended incrementally. The second term on the right-hand side is the reinvestment rate for existing benefits. The difference between the two is the marginal reduction in the value of the existing project benefits from extending the life of the project incrementally beyond T. The condition for the MIRR to increase is that the percentage increase in the level of benefits must exceed the net percentage decrease in the value of existing benefits due to additional discounting.

Further discussion of this condition is intriguing. Note first that the right-hand side is positive in discussions of interest since a negative value would indicate that the MIRR through time T is less than the reinvestment rate and thus it is suboptimal to invest in the project. Second, note that the inequality is not universally valid—there are cases where it will hold, and cases where it will not. Third, note that as $b_{t+1} \to 0$, so does the left-hand side, so that the inequality does not hold. That is, it is always possible to find a strongly dominant project with a lower MIRR. More formally, we have shown that for any project P_1 with $\mathrm{MIRR}(P_1) > \delta^r$, there is a project P_1^* such that P_1^* s.d. P_1 and $\mathrm{MIRR}(P_1^*) < \mathrm{MIRR}(P_1)$.